Feinberg-Pais Equation for the Trace of Ladder Diagrams*

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The equations of Feinberg and Pais for the trace of the sum of ladder diagrams in lepton scattering are examined by summing Feynman diagrams, and also in configuration space. Both methods establish the truth of a conjecture made by Feinberg and Pais: that the equation is equivalent to one which may be iterated without generating divergent integrals.

1. INTRODUCTION

In this paper we discuss the equation for the trace of the sum of uncrossed ladder diagrams written down by Feinberg and Pais as part of their peratization program.[1] This equation is investigated in two complementary ways, each of which avoids the iteration scheme of F. P. II, whose convergence has not been demonstrated.

In Sec. 2 the equation is considered in momentum space and its perturbation solution is written down. Although the peratization program is intended to improve on perturbation theory there is no reason to suppose that this improvement cannot be expressed in terms of manipulations of the perturbation series. This turns out to be the case. The series can be written as a sum of terms which are convergent as M^2 (the regulator mass) tends to infinity, together with the ratio of two expressions each of which is a series whose terms become infinite as $M^2\to\infty$. Their large M^2 behavior can be investigated by methods developed for discussing the high-energy behavior of Feynman integrals.[2,3] It is found that, at least for small values of the coupling constant, the numerator tends to infinity less rapidly with M^2 than the denominator. It is concluded therefore that the sum of convergent terms gives the correct peratized solution, which confirms a conjecture of Feinberg and Pais.

In Sec. 3 the integral equation in momentum space is turned into a differential equation in configuration space in the limit in which M^2 has gone to infinity. The solutions of this differential equation are then discussed by Green's function methods. The equation for T^+ has a perfectly well-defined solution and the equation for T^- may be given a well-defined solution by a limiting procedure which is closely connected with simulating the limit $M^2\to\infty$. Again both solutions obtained correspond to those conjectured by Feinberg and Pais, and are solutions of a related integral equation which can be solved by expansion in the coupling constant (for not too-high energies).

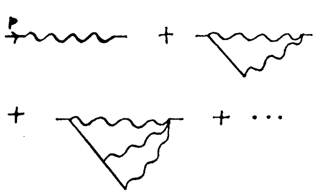

FIG. 1. The diagrams for $T(p)$.

2. MOMENTUM-SPACE PERTURBATION THEORY

The equations given in F. P. II for the trace are

$$T^{\pm}(p)=-\frac{ig^2(4+m^{-2}p^2)}{p^2+m^2}\pm\frac{4ig^2M^4}{(2\pi)^4}\times\int\frac{d^4p'[4+m^{-2}(p-p')^2]T^{\pm}(p')}{(p'^2+\mu^2)(p'^2+M^2)^2[(p'-p)^2+m^2]},\quad(1)$$

where we have reinserted the lepton mass μ since this adds no further complication in the cases of $e\nu_e$ and $\mu\nu_\mu$ scattering. We investigate the iterative solutions of these equations. They are given by the set of diagrams of Fig. 1 where the rules of interpretation are: (i) The wavy lines correspond to a propagator

$$(4+m^{-2}q^2)/(q^2+m^2);\quad(2)$$

(ii) the solid lines correspond to a propagator

$$M^4/(q^2+\mu^2)(q^2+M^2)^2,\quad(3)$$

(iii) the diagram with n loops is multiplied by a factor

$$-ig^2[\pm4ig^2/(2\pi)^4]^n.\quad(4)$$

The propagator (2) may now be written as

$$(1/m^2)+(3/q^2+m^2),\quad(5)$$

a decomposition which we denote graphically by a wavy line equal to a dot plus broken line. The effect of this decomposition on a typical graph is shown in Fig. 2. The effect of a dot is to decompose the graph into two subgraphs joined only through the dot. The contribution of a graph is just the product of the contribution of the separate subgraphs into which it is divided by the appearance of dots. This observation immediately

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[1] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963); **133**, B477 (1964). Referred to as F. P. I and II, respectively. We use the notation of these papers where appropriate.

[2] J. C. Polkinghorne, J. Math. Phys. **4**, 503 (1963), P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) **22**, 263, 199 (1963), I. G. Halliday, Nuovo Cimento **30**, 177 (1963).

[3] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963), T. L. Trueman and T. Yao, Phys. Rev. **132**, 2741 (1963). J. C. Polkinghorne, J. Math. Phys. (to be published).

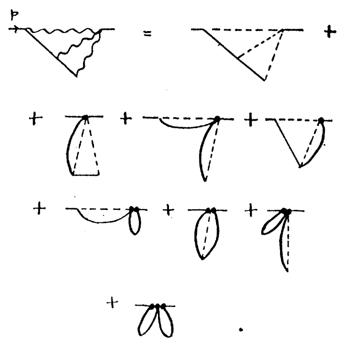

FIG. 2. The decomposition of a typical diagram.

enables a summation to be performed to yield

$$T^{\pm}(p)=C^{\pm}(p)-\frac{ig^2}{m^2}\frac{G^{\pm}(p)G^{\pm}(0)}{1-F^{\pm}}, \qquad (6)$$

where the functions $C^{\pm}$, $G^{\pm}$, $F^{\pm}$ are given by the sum of contributions corresponding to the series of graphs illustrated in Figs. 3, 4, and 5, subject to the rule (iii) being replaced in the evaluation of F and G by the multiplication of a diagram with n loops by a factor

$$[\pm 4ig^2/(2\pi)^4]^n. \qquad (7)$$

The terms appearing in C are all convergent, even when the limit $M^2\to\infty$ is taken. They correspond to the iterative solution of the equation stated in F. P. II following Eq. (2.37). We shall show in fact that the second term in (6) tends to zero as $M^2\to\infty$, at least for sufficiently small values of the coupling constant, and so confirm the conjecture of Feinberg and Pais that $C^{\pm}(p)$ is the peratized solution for the trace.

FIG. 3. The diagrams for C.

In order to investigate the behavior for large M^2 of the integrals appearing in the series expansions of $G^{\pm}$ and $F^{\pm}$ it is convenient to use the methods which have been developed for investigating the high-energy behavior of perturbation theory.[2,3] An account of how these methods may be adapted to the present problem is contained in Appendix A. The principal results are that, in the leading approximation,

$$G^{\pm}\sim(M^2)^{\lambda_{\pm}}, \qquad (8)$$

where

$$\lambda_{\pm}=\mp 3g^2/4\pi^2, \qquad (9)$$

and that

$$F^{\pm}\sim M^2+O[(M^2)^{\lambda_{\pm}}]. \qquad (10)$$

Thus as $M^2\to\infty$ for small g^2 the second term of (6) tends to zero.

$T^{\pm}(p)$ originally appeared in an integral equation in which the parameter p was integrated over an infinite range. It is of interest, therefore to consider the behavior of $C^{\pm}(p)$ at high p^2. Standard methods[2] applied to the function obtained by allowing $M^2\to\infty$ in $C^{\pm}(p)$ yield

$$C^{\pm}\sim(p^2)^{-1+\lambda_{\pm}}, \quad p^2\to\infty \qquad (11)$$

in the leading approximation. For C^+ this makes the original integral manifestly convergent even when M^2 is put equal to infinity in the kernel. For C^- a regulator must be retained in the kernel to give it meaning. This corresponds to a behavior found also in configuration space which is discussed in the following section.[4]

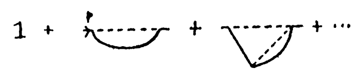

FIG. 4. The diagrams for G.

3. THE TRACE EQUATION IN CONFIGURATION SPACE

Defining

$$S^{\pm}(p)=(p^2+\mu^2-i\epsilon)^{-1}T^{\pm}(p) \qquad (12)$$

and the Fourier transform

$$S^{\pm}(x)=(2\pi)^{-4}\int d^4p S^{\pm}(p)\exp(ip\cdot x),$$

Eq. (1) (with the limit $M^2\to\infty$ already taken) may be written

$$(\Box-\mu^2)S^{\pm}(x)\pm 12ig^2\triangle_F(x)S^{\pm}(x) = 3ig^2\triangle_F(x)+iA^{\pm}\delta^4(x) \qquad (13)$$

where

$$A^{\pm}=(g^2/m^2)[1\mp 4S^{\pm}(0)]. \qquad (14)$$

We consider Eq. (13) for spacelike x, and define $r=+(x^2)^{1/2}$. Equations for timelike x may be obtained by the substitution $r\to e^{i\pi}s$. Eq. (13) becomes

$$\frac{d^2S^{\pm}}{dr^2}+3r^{-1}\frac{dS^{\pm}}{dr}-\mu^2S^{\pm}\pm 12ig^2\triangle_F S^{\pm} = iA^{\pm}\delta^4(x)+3ig^2\triangle_F. \qquad (15)$$

Since for small x^2

$$\triangle_F\sim-(4\pi^2 i)^{-1}[x^{-2}-i\pi\delta(x^2)], \qquad (16)$$

the solutions of the homogeneous part of (15) go as

$$(r^2+i\epsilon)^{-\frac{1}{2}+\frac{1}{2}\nu} \text{ or } (r^2+i\epsilon)^{-\frac{1}{2}-\frac{1}{2}\nu} \qquad (17)$$

at the origin, where

$$\nu=(1\pm 3g^2/\pi^2)^{1/2}. \qquad (18)$$

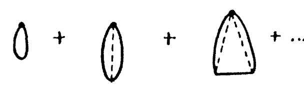

FIG. 5. The diagrams for F.

[4] Since this paper was written, we have seen a paper by Y. Pwu and T. T. Wu [Phys. Rev. **133**, B778 (1964)] which also uses x-space methods. They treat the cutoff more carefully than we do in Sec. 3 but calculate only $\lim_{M^2\to\infty}\lim_{p^2\to 0} T(p)$. They also determine the corresponding limit of $\beta(p)$, defined in F. P. II, but difficulties arise in attempting to use our methods to evaluate $\lim_{M^2\to\infty}\beta(p)$.

The boundary conditions appropriate to (15) are that there should be an $\exp(-\mu r)$ behavior at infinity [corresponding to the pole in (12)], and that $S^{\pm}(0)$ should exist. The latter requirement can be met by the choice of the first term in (17) for S^{+}, because ν is then greater than one. We shall see later that S^{-} may be defined by a limiting procedure, provided that the first term in (16) is chosen here also.

We therefore define two solutions $U^{\pm}$ and $V^{\pm}$ of the homogeneous part of (15) by the conditions that

$$U^{\pm}\sim r^{-1+\nu} \text{ as } r\rightarrow 0,$$

and that $V^{\pm}$ has an $\exp(-\mu r)$ behavior for large r and is normalized so that

$$V^{\pm}\sim r^{-1-\nu} \text{ as } r\rightarrow 0.$$

The appropriate Green's function is then $-G^{\pm}(r,r')r'^3/2\nu$, where

$$\begin{aligned}G^{\pm}(r,r')&=U^{\pm}(r)V^{\pm}(r') \text{ for } r\leqslant r'\\&=V^{\pm}(r)U^{\pm}(r') \text{ for } r\geqslant r'.\end{aligned}\qquad(19)$$

In terms of this function, the solutions of (15) are[5]

$$S^{\pm}(r)=+(A^{\pm}/4\pi^2\nu)G^{\pm}(r,0)-(3ig^2/2\nu)\times\int_0^\infty G(r,r')\triangle_F(r')r'^3dr'.\qquad(20)$$

Putting $r=0$ in the last equation and using (14), we may solve formally for $A^{\pm}$, to obtain

$$A^{\pm}=[G^{\pm}(0,0)/\pi^2\nu\pm m^2/g^2]^{-1}\times\left[\pm1+(6g^2i/\nu)\int_0^\infty G^{\pm}(0,r')\triangle_F(r')r'^3dr'\right].\qquad(21)$$

Equations (20) and (21) constitute the formal solution of the problem, and are the x-space equivalent of Eq. (6).

We must now discuss S^{+} and S^{-} separately. In the case of S^{+}, $G^{+}(0,r')$ exists, and in fact

$$\begin{aligned}&\int_0^\infty G^{+}(0,r')\triangle_F(r')r'^3dr'\\&=\lim_{\delta\to0}\left[\delta^{-1+\nu}\int_\delta^\infty\triangle_F(r')r'^{2-\nu}dr'+\delta^{-1-\nu}\int_0^\delta\triangle_F(r')r'^{2+\nu}dr'\right]\\&=-\nu/(6g^2i),\end{aligned}\qquad(22)$$

where to obtain the last expression we have used (18). Inserting (22) into (21), we see that A^{+} vanishes identically. This conclusion for A^{+} depends upon the particular structure of Eq. (1): the similarity between the kernel and the inhomogeneous term.

[5] The formula

$$\int_0^\infty f(r)r^3\delta^4(x)dr=-\frac{i}{2\pi^2}f(0)$$

is required.

We now turn to the solution for S^{-}. Neither $G^{-}(0,r')$ nor $G^{-}(r,0)$ exist. We therefore define a quantity $A^{-}(\delta)$ by replacing $G^{-}(0,r')$ by $G^{-}(\delta,r')$ and $G^{-}(0,0)$ by $G^{-}(\delta,\delta)$ in (21). We also define $S^{-}(r,\delta)$ by substituting $A^{-}(\delta)G^{-}(r,\delta)$ for $A^{-}G^{-}(r,0)$ in (20). From (19) a short consideration shows that $A^{-}(\delta)G^{-}(r,\delta)$ goes to zero as $\delta^{2\nu}$ as $\delta\rightarrow0$, because the infinite terms in the numerator are over-compensated by $G^{-}(\delta,\delta)\sim\delta^{-2}$ in the denominator [this being the x-space analog of Eq. (10)].

We are thus lead to define

$$S^{-}(r)=\lim_{\delta\to0}S^{-}(r,\delta)$$

to be the solution. This definition corresponds to letting the cutoff become infinite only after the equation is solved. The S^{-} thus defined is given by Eq. (20) with the term in A^{-} omitted.

The solutions for S^{+} and S^{-}, although they have a different logical status, are each solutions of (15) with the term containing $A^{\pm}$ omitted. This is an equation which can be solved by straightforward iteration. We have thus proved the conjecture made by F. P. II in the paragraph following Eq. (2.37), and therefore confirmed the conclusions (2.29) and (2.32).

ACKNOWLEDGMENT

We wish to thank I. G. Halliday for a valuable discussion.

APPENDIX A

We consider the diagram of Fig. 6 which contributes to $G(p)$. Each solid line has two Feynman parameters associated with it, an x for the $(q^2+M^2)^{-2}$ factor and an α for the $(q^2+m^2)^{-1}$ factor. The contribution corresponding to Fig. 6 is

$$I=(M^2)^{2r}\left(\mp\frac{3g^2}{4\pi^2}\right)^r\Gamma(2r)\times\int_0^1 d\xi\frac{\delta(\sum\xi-1)\prod_{i=1}^r x_iC(\xi)^{2r-2}}{[D(\xi;p^2,M^2)]^{2r}},\qquad(\text{A1})$$

where ξ is a collective symbol for the x_i, α_i, β_i, and C and D are the Feynman numerator and denominator functions associated with the diagram obtained from Fig. 6 by replacing each solid line by two consecutive lines, one corresponding to mass M and parameter x and the other corresponding to mass m and parameter c. The structure of D is

$$D=-(\sum_{i=1}^r x_i)C(\xi)M^2-\delta(p^2,\xi).\qquad(\text{A2})$$

If we use the Mellin transform method[3] for obtaining the high M^2 behavior, it is convenient to rewrite (A1) in the form

$$I_r' = I/(M^2)^{2r}(\mp 3g^2/4\pi^2)$$
$$= \int_0^\infty d\xi \prod_{i=1}^{r} x_i C^{-2} \exp\left[-\sum_{i=1}^{n} x_i M^2 - \delta/C\right]. \quad \text{(A3)}$$

We now take the Mellin transform of I' with respect to M^2, denoting the transform by $\tilde{I}_r'(\alpha)$. The result is

$$\tilde{I}_r'(\alpha) = \Gamma(-\alpha) \int_0^\infty d\xi \prod_{i=1}^{r} x_i \Big(\sum_{i=1}^{r} x_i\Big)^{\alpha} C^{-2} e^{-\delta/C}. \quad \text{(A4)}$$

This integral is divergent when $\alpha=-2r$. In order to remove the divergencies and continue below $\alpha=-2r$ it is necessary to perform certain scalings and then integrate by parts. The scalings are given by

$$\alpha_i = \rho_1 \alpha_i^{(1)},\ \beta_i = \rho_1 \beta_i^{(1)},$$
$$x_i = \rho_1 x_i^{(1)},\quad i \leqslant r;$$
$$\alpha_i^{(s-1)} = \rho_s \alpha_i^{(s)},\ \beta_i^{(s-1)} = \rho_s \beta_i^{(s)},\ i \leqslant r-s+1, \quad \text{(A5)}$$
$$x_i^{(s-1)} = \rho_s x_i^{(s)},\quad i \leqslant r;$$
$$x_i^{(r)} = \rho_{r+1} x_i^{(r+1)},\quad i \leqslant r.$$

After the integrations by parts have been performed we obtain

$$\tilde{I}_r'(\alpha) = \frac{\Gamma(-\alpha)}{(2r+\alpha)^{r+1}} \int_0^\infty d\rho_1 \cdots d\rho_{r+1} d\alpha_r^{(1)} d\beta_r^{(1)} \cdots$$
$$\times d\alpha_1^{(r)} d\beta_1^{(r)} dx_i^{(r+1)} \prod_{i=1}^{r+1} \rho_i^{2r+\alpha} \prod_{i=1}^{r} x_i^{(r+1)}$$
$$\times \Big(\sum_{i=1}^{r} x_i^{(r+1)}\Big)^{\alpha} (-)^{r+1} \frac{\partial^{r+1}}{\partial\rho_1 \cdots \partial\rho_{r+1}}$$
$$\times \Big[\prod_{i=1}^{r} \delta\big(\alpha_i^{(r-i+1)} + \beta_i^{(r-i+1)} + \rho_{i+1} - 1\big)$$
$$\times \delta\Big(\sum_{i=1}^{r} x_i^{(r+1)} - 1\Big) \tilde{C}^{-2} \exp(-\rho_1 \tilde{\delta}/\tilde{C})\Big]. \quad \text{(A6)}$$

The expression in square brackets in (A6) is written in a highly symbolic form. The δ functions represent the fact that in order to integrate by parts certain variables must be eliminated (say the $\beta_i^{(r-i+1)}$) and replaced by the expressions given by the δ functions. The functions $\tilde{C}$ and $\tilde{\delta}$ are formed from C and δ by making the scaling substitutions (A5) and extracting a factor ρ_s from C' for each scaling, and a factor ρ_1^2 from δ for the first scaling, and ρ_s for subsequent scalings.

The form (A6) explicitly exhibits the pole of order $(r+1)$ at $\alpha=-2r$. If we just evaluate the leading behavior (that is, the highest power of the logarithm) it is only necessary to find the residue of this pole. This is given by

$$\Gamma(2r) \int_0^\infty d\alpha_r^{(1)} d\beta_r^{(1)} \cdots d\alpha_1^{(r)} d\beta_1^{(r)} dx_i^{(r+1)}$$
$$\times \prod_{i=1}^{r} \delta\big(\alpha_i^{(r-i+1)} + \beta_i^{(r-i+1)} - 1\big)$$
$$\times \prod x_i^{(r+1)} \delta\Big(\sum_{i=1}^{r} x_i^{(r+1)} - 1\Big) = 1. \quad \text{(A7)}$$

Thus

$$I_r^{\pm} \Big(\mp \frac{3g^2}{4\pi^2}\Big)^r \frac{(\ln M^2)^r}{\Gamma^{(r+1)}}. \quad \text{(A8)}$$

This gives

$$G^{\pm} \sim (M^2)^{\lambda_{\pm}}, \quad \text{(A9)}$$

where

$$\lambda_{\pm} = \frac{\mp 3g^2}{4\pi^2}. \quad \text{(A10)}$$

It is also possible to find the residues of the lower-order poles by methods analogous to those used in the discussion of the high energy behavior of ladder diagrams.[3] Unlike that case, however, a complete summation of the series has not been obtained and so we shall not go into details. The partial summations which have been made alter (A9) and (A10) in two ways, neither of which appears important for peratization. Extra terms appear in the expansion (A10) corresponding to higher powers of g, and the pole in α is turned into branch cut with the effect of producing factors of $(\ln M^2)$ in (A9).

The integrals corresponding to graphs of type Fig. 7 contribute to F. They may be analyzed in a similar way to that described above. The leading behavior turns out to be given by

$$\frac{M^2}{m^2}(\mp 3g^2/4\pi^2)^r$$
$$\times \left[\Gamma(2r+1) \int_0^1 \frac{\prod_{i=1}^{r} x_i \delta(\sum \xi - 1)}{C^2(\xi)\big(\sum_{i=1}^{r} x_i\big)^{2r+1}}\right]. \quad \text{(A11)}$$

In addition each integral contains terms behaving like $(\ln M^2)^s$ $(s \leqslant r+1)$. These terms are proportional to μ^2, the lepton mass. The leading $(\ln M^2)^{r+1}$ terms sum to give

$$(\mu^2/m^2)(M^2)^{\lambda_{\pm}}. \quad \text{(A12)}$$

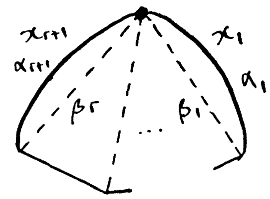

FIG. 7. A typical F diagram.

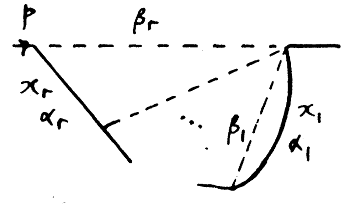

FIG. 6. A typical G diagram.